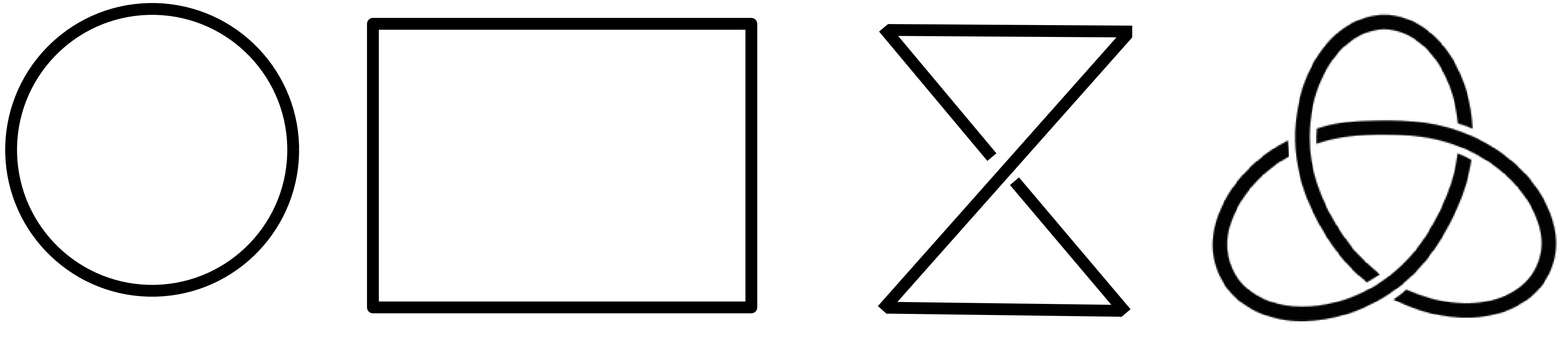
Topological Quantum Computing
What gives?

Henrique Ennes
CEMRACS 2025
05/08/2025

Knots

Knots are embeddings of the circle in $\mathbb{R}^3$.



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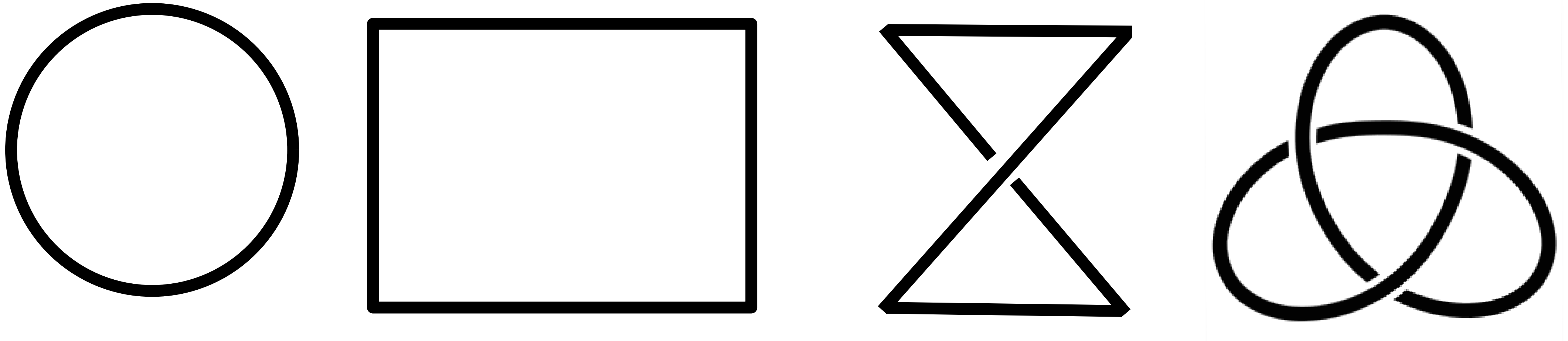

Two knots will be called *isotopic* if we can bring one to the other without tearing them apart.

ISOTOPIC

NON-ISOTOPIC

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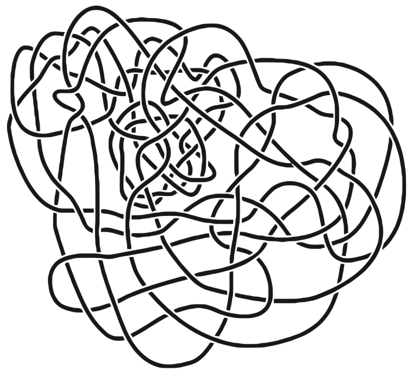
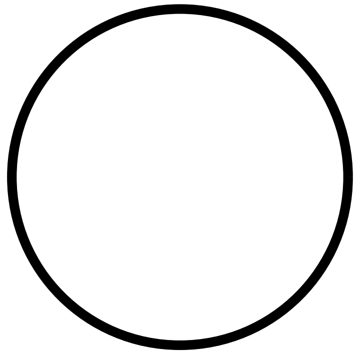
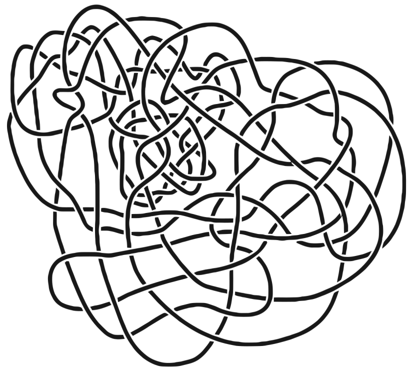
Knots

The problem of telling isotopic knots apart is computationally very hard.

PROBLEM			STATUS
	“ = ”		NP ∩ coNP
	“ = ”		DECIDABLE

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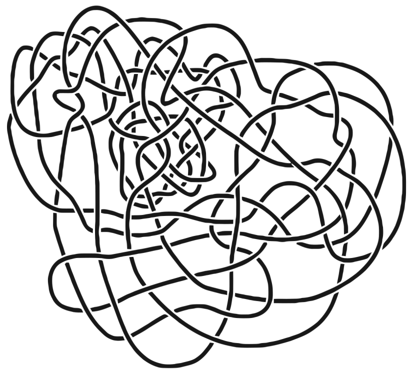
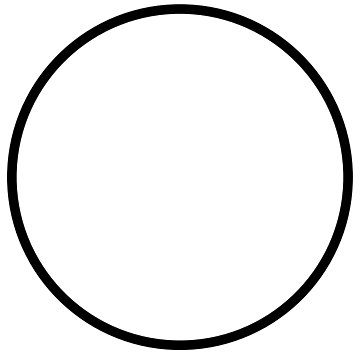
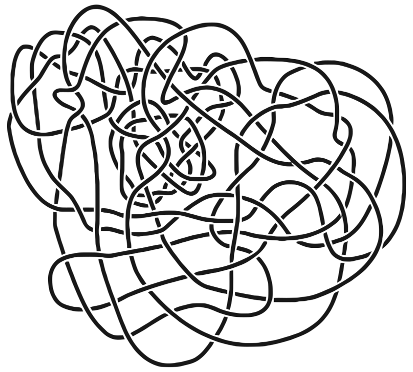
	" = "		NP $\cap$ coNP
	" = "		DECIDABLE

We can use **invariants** to get approximations to this problem

$$K \text{ is isotopic to } K' \implies \langle K \rangle = \langle K' \rangle$$

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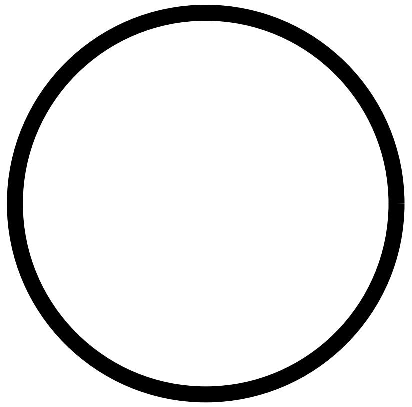
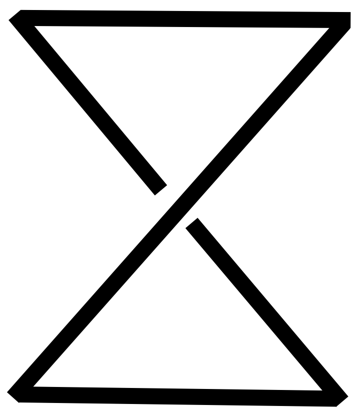
Knots

Polynomials give a nice list of invariants

$V_t = \Bigg($  $\Bigg) = 1$ $\quad$ $V_t = \Bigg($  $\Bigg) = 1$

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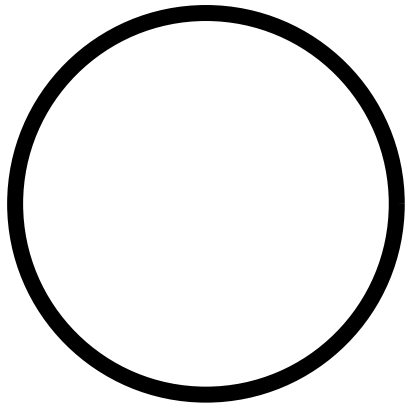
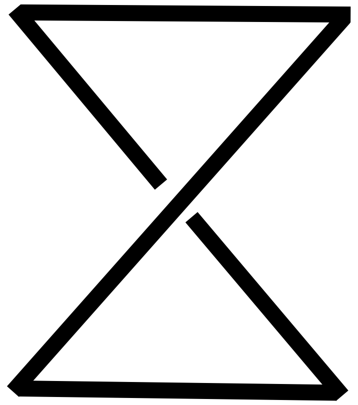
$V_t = \Big($  $\Big) = 1$ $\qquad V_t = \Big($  $\Big) = 1$

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Theorem (Vertigan, Kuperberg):
Computing (or even well-approximating) the Jones polynomial at some values of t is #P-hard.

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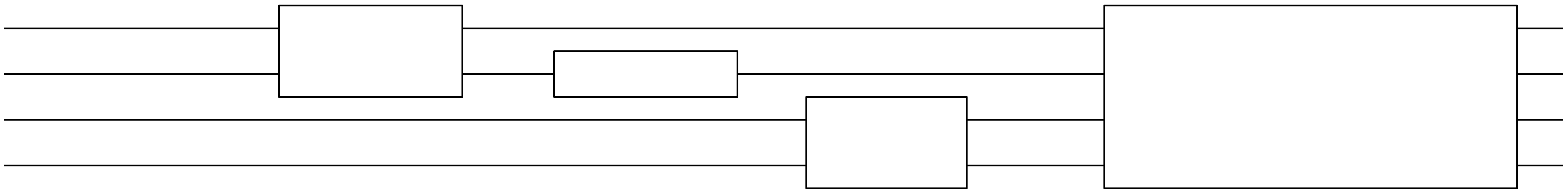
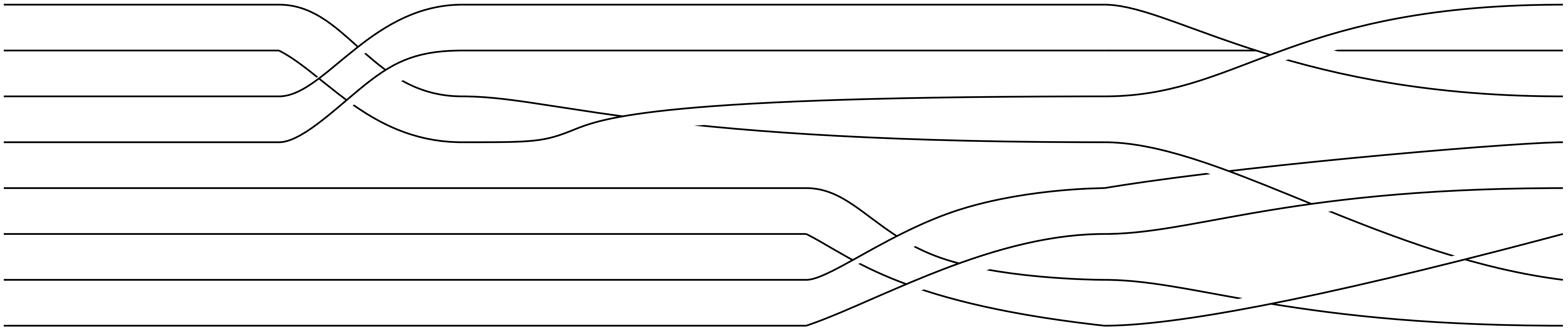
And where is quantum?





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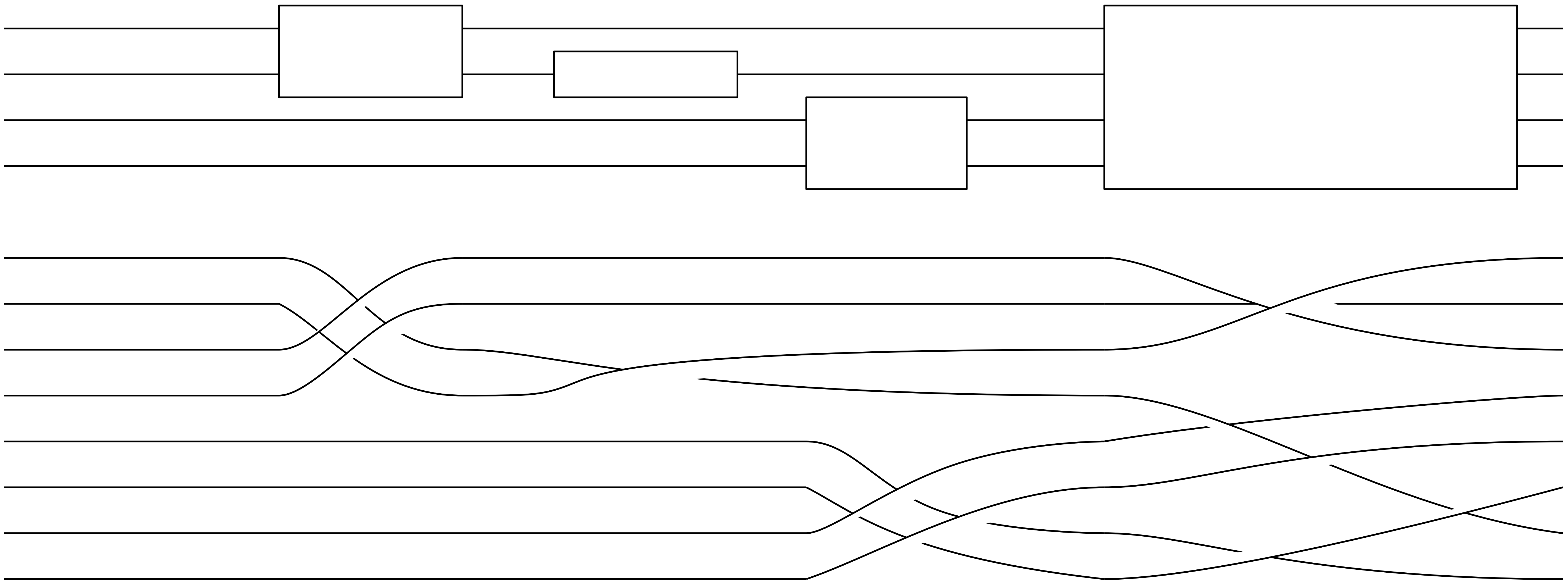
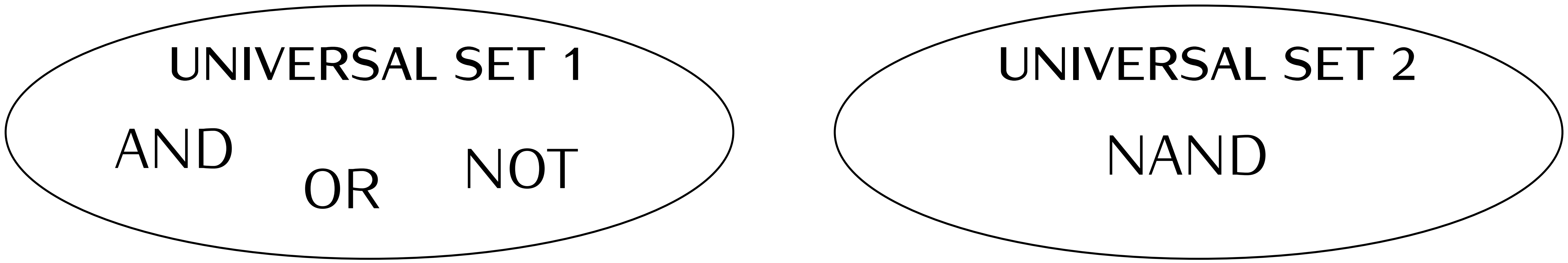


$$|\langle 0^{\otimes n}|C|0^{\otimes n}\rangle|^2 \approx |V_t(L)|^2/|t^{1/2}+t^{-1/2}|^{4n}$$

Solovay-Kitaev Theorem

We call a set of classical gates $\mathcal{G}$ **universal** if every Boolean function can be written using it.



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UNIVERSAL SET 1

AND OR NOT

UNIVERSAL SET 2

NAND

We call a set of quantum gates $\mathcal{G}$ **universal** if every matrix $U \in \mathrm{PU}(4)$ can be **approximated** up to any error ϵ using it.

UNIVERSAL SET 1

H CNOT $\pi/8$

UNIVERSAL SET 2

TOFFOLI X H

Solovay-Kitaev Theorem

Solovay-Kitaev Theorem

Let $\mathcal{G} = \{G_1, \ldots, G_g, G_1^{-1}, \ldots, G_g^{-1}\}$ be a finite set of matrix generators such that $\langle \mathcal{G} \rangle$ is dense in PU(4). Then

- $\mathcal{G}$ is universal;
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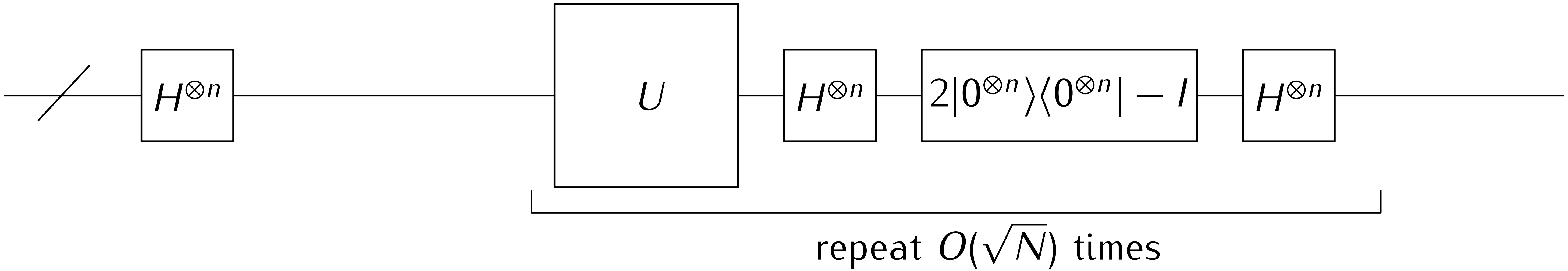
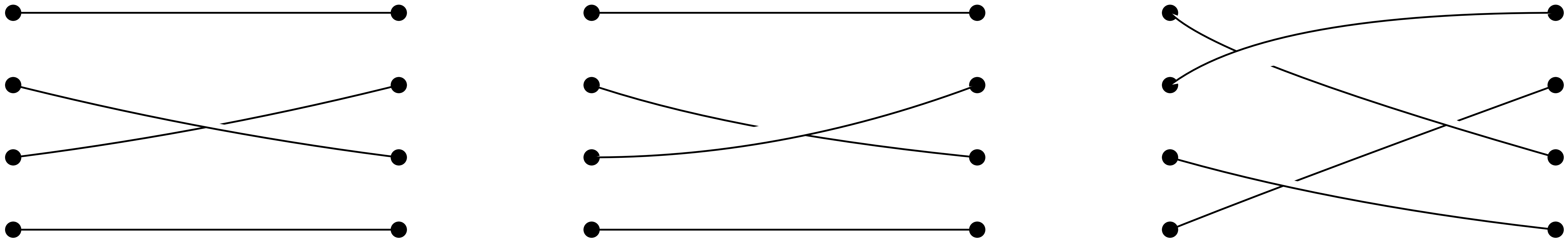


By changing the gates to $\mathcal{G}$, we need only $O(\sqrt{N}\log^c(N/\epsilon))$ gates for an error of at most ϵ instead of $O(N/\epsilon)$ =CLASSICAL TIME.

A group for knots

Let us consider the **braids** of n strands, B_n



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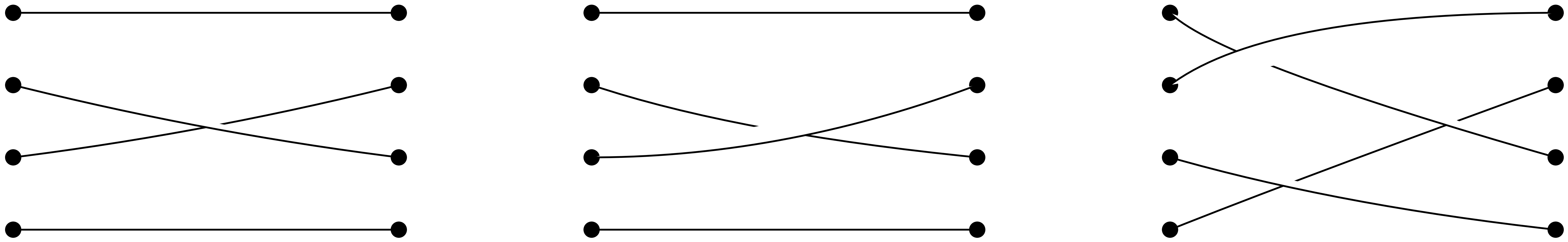
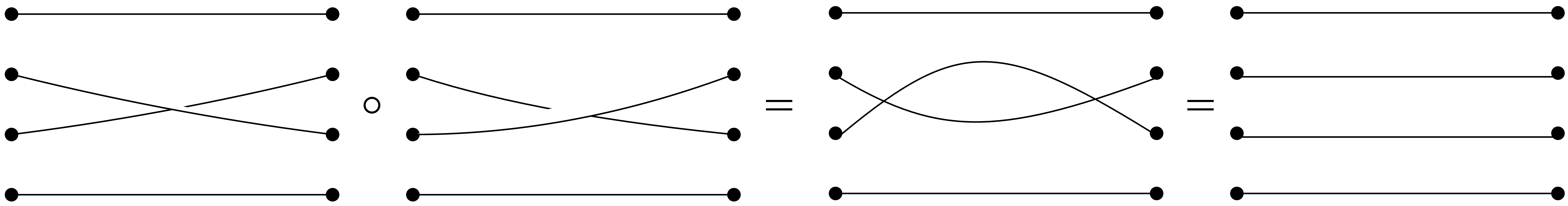


We can consider the composition of braids



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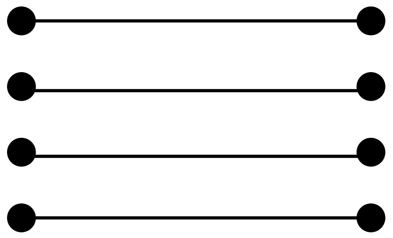
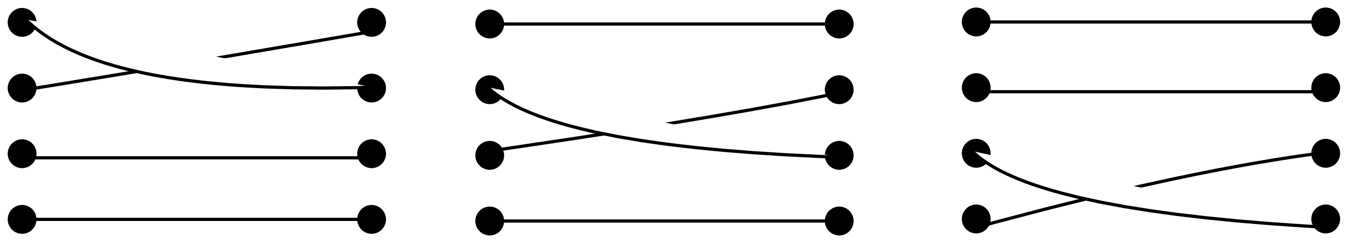
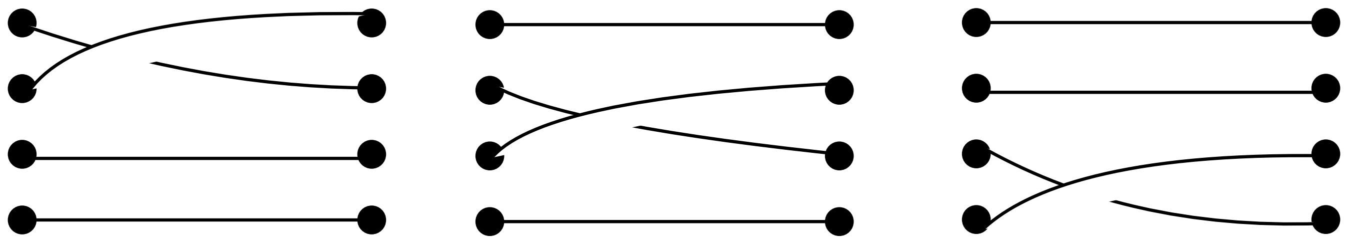
The n braid group B_n has

- identity 
- generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ 
- inverses 

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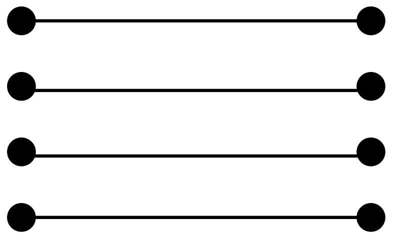
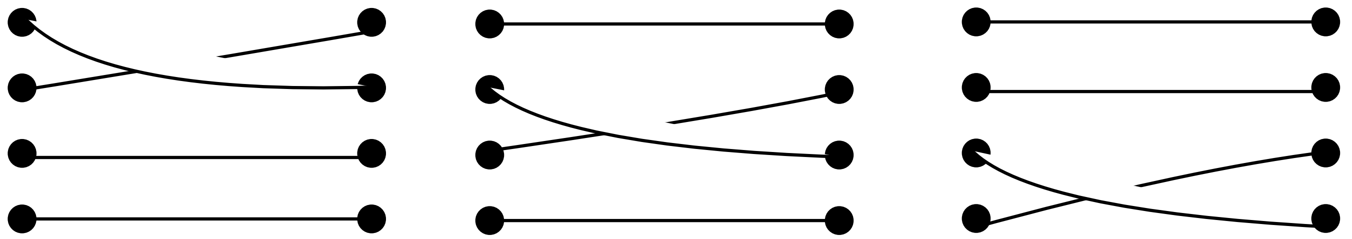
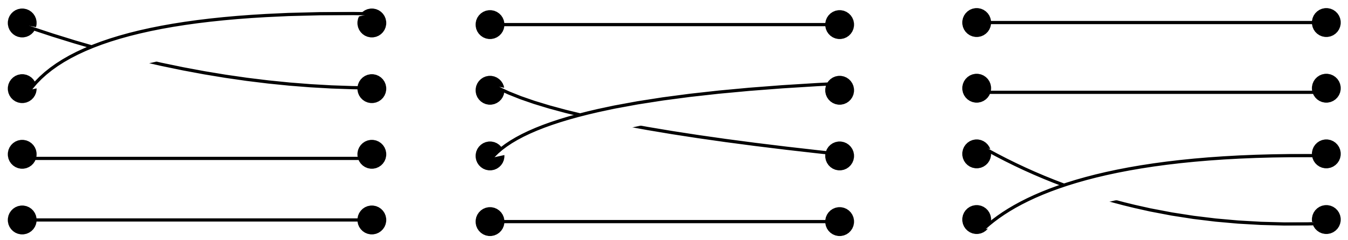
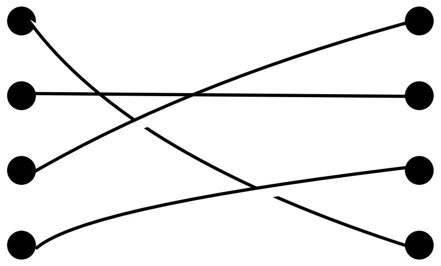
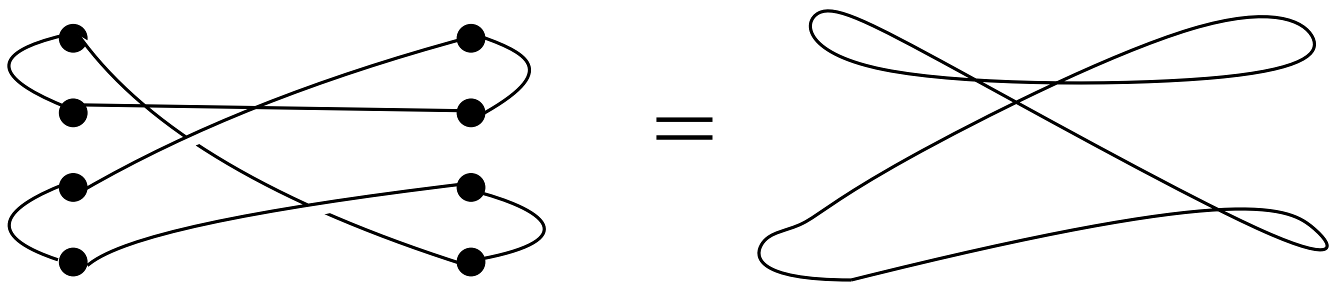
- identity 
- generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ 
- inverses 

Every knot can be obtained by closing braids





$b \in B_n$

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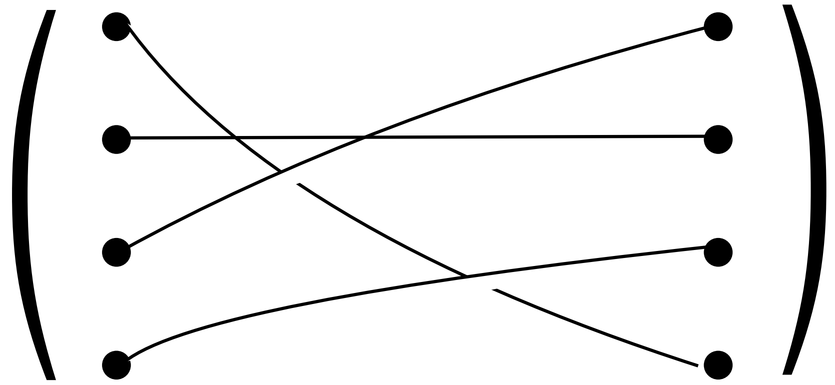
The **(unitary) Jones representations** of the braid groups B_n at value t are homomorphisms $\rho_{n,t} : B_n \to \mathrm{PU}(n')$

$\rho_{n,t}$  $= U$

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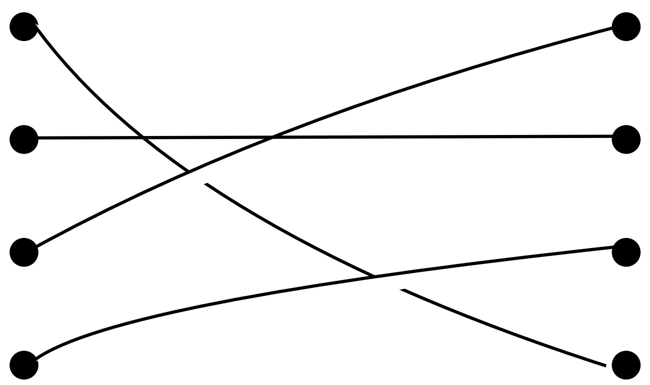
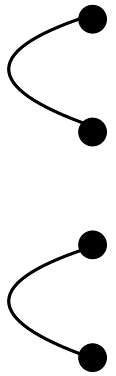
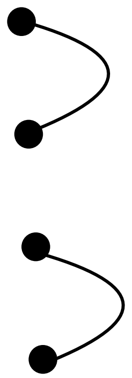
$\rho_{n,t}\Big($  $\Big) = U$

If we vectorize the closure

 $\mapsto \langle 0| \in (\mathbb{C}^{n'})^*$  $\mapsto |0\rangle \in \mathbb{C}^{n'}$

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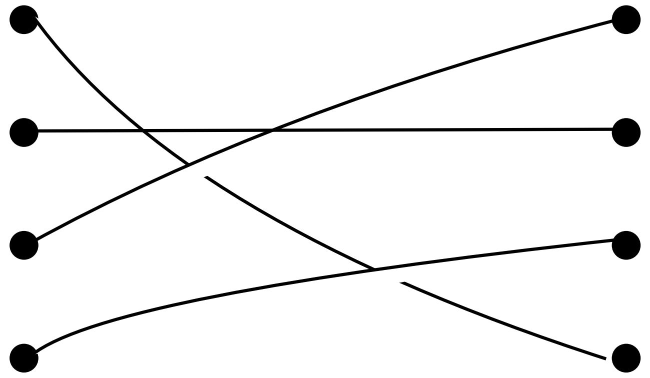
$\rho_{n,t}\Big($  $\Big) = U$

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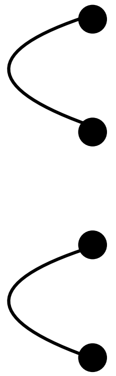
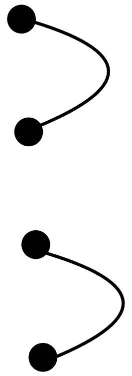
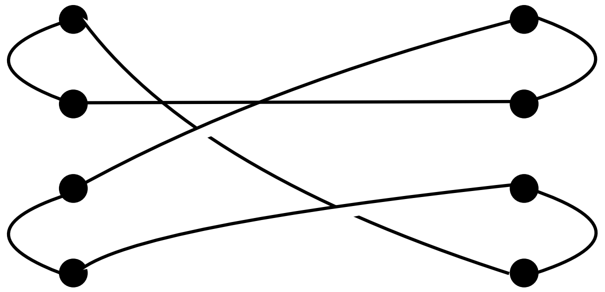
 $\mapsto \langle 0| \in (\mathbb{C}^{n'})^*$  $\mapsto |0\rangle \in \mathbb{C}^{n'}$

we have Jones polynomials at t (up to a factor)

$V_{n,t}\Big($  $\Big) = \frac{1}{-(t^{1/2}+t^{-1/2})t^c}\langle 0|U|0\rangle$

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Topological quantum computing

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When t is certain roots of the unity, $\rho_{n,t}(B_n)$ is dense in $\mathrm{PU}(n')$ for all $n \geq 4$.

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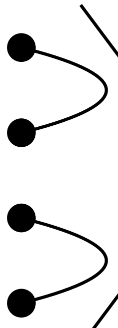
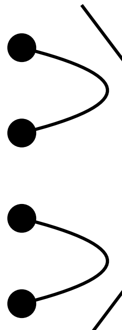
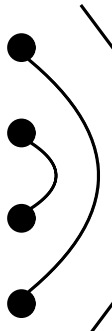
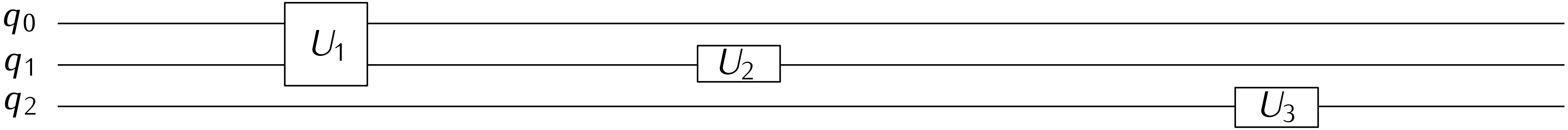
1. Represent $\mathbb{C}^2$ with a basis

$|0\rangle \mapsto \frac{1}{t^{1/2}+t^{-1/2}}$ |  $\rangle$

$|1\rangle \mapsto \frac{1}{\sqrt{(t+1+t^{-1})(t^{1/2}+t^{-1/2})}}$ |  $\rangle$ $+ \frac{1}{\sqrt{(t+1+t^{-1})(t^{1/2})}}$ |  $\rangle$

Topological quantum computing

2. Apply Solovay-Kitaev to represent each gate of the circuit with $\rho_{4n,t}(\sigma_i)$



Solovay-Kitaev's output

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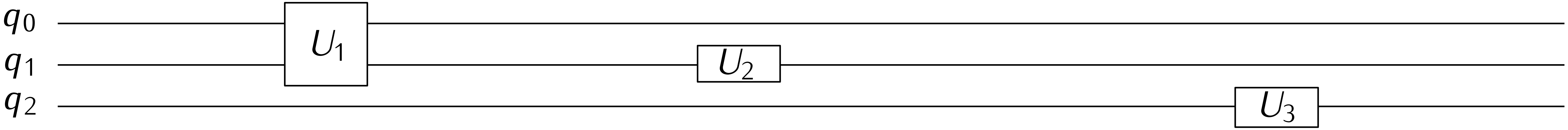

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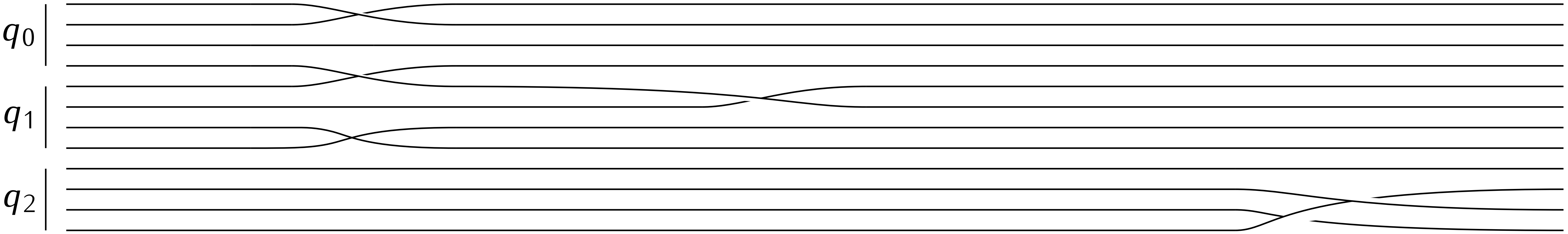

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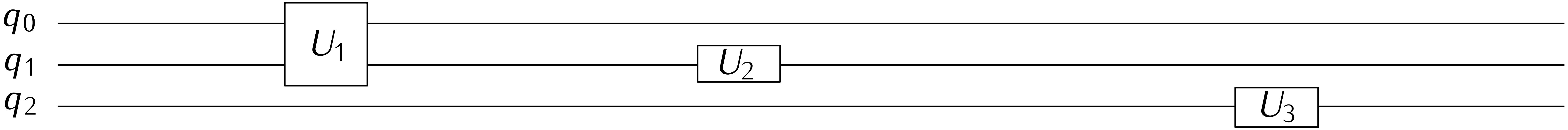

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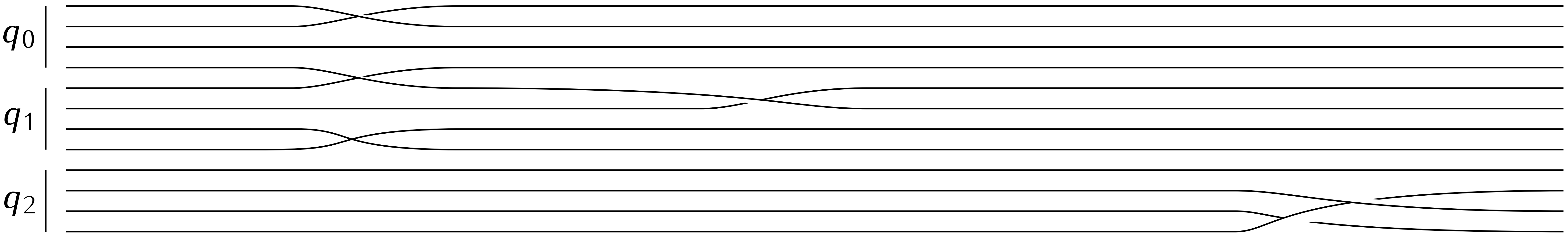

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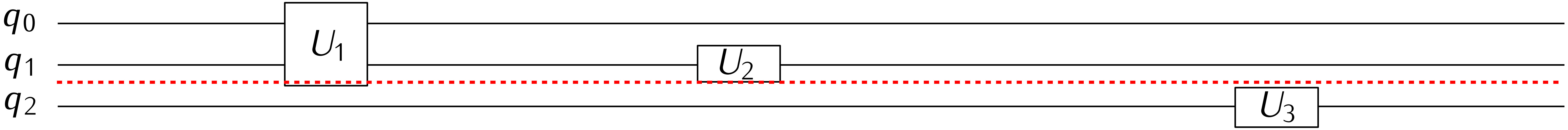

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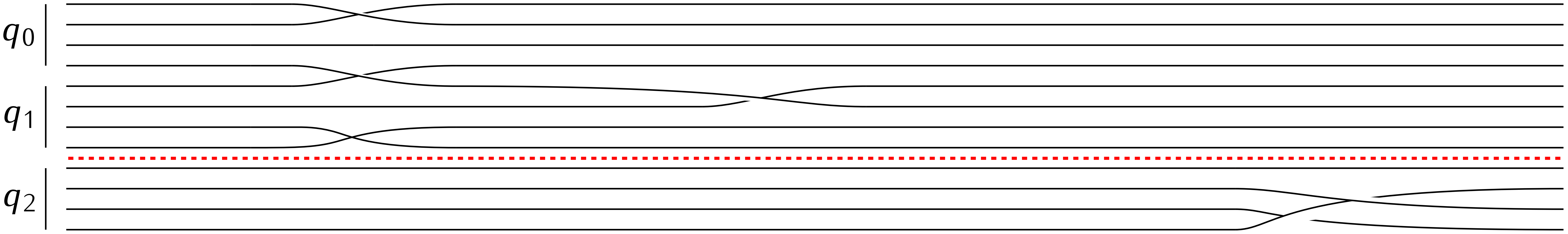

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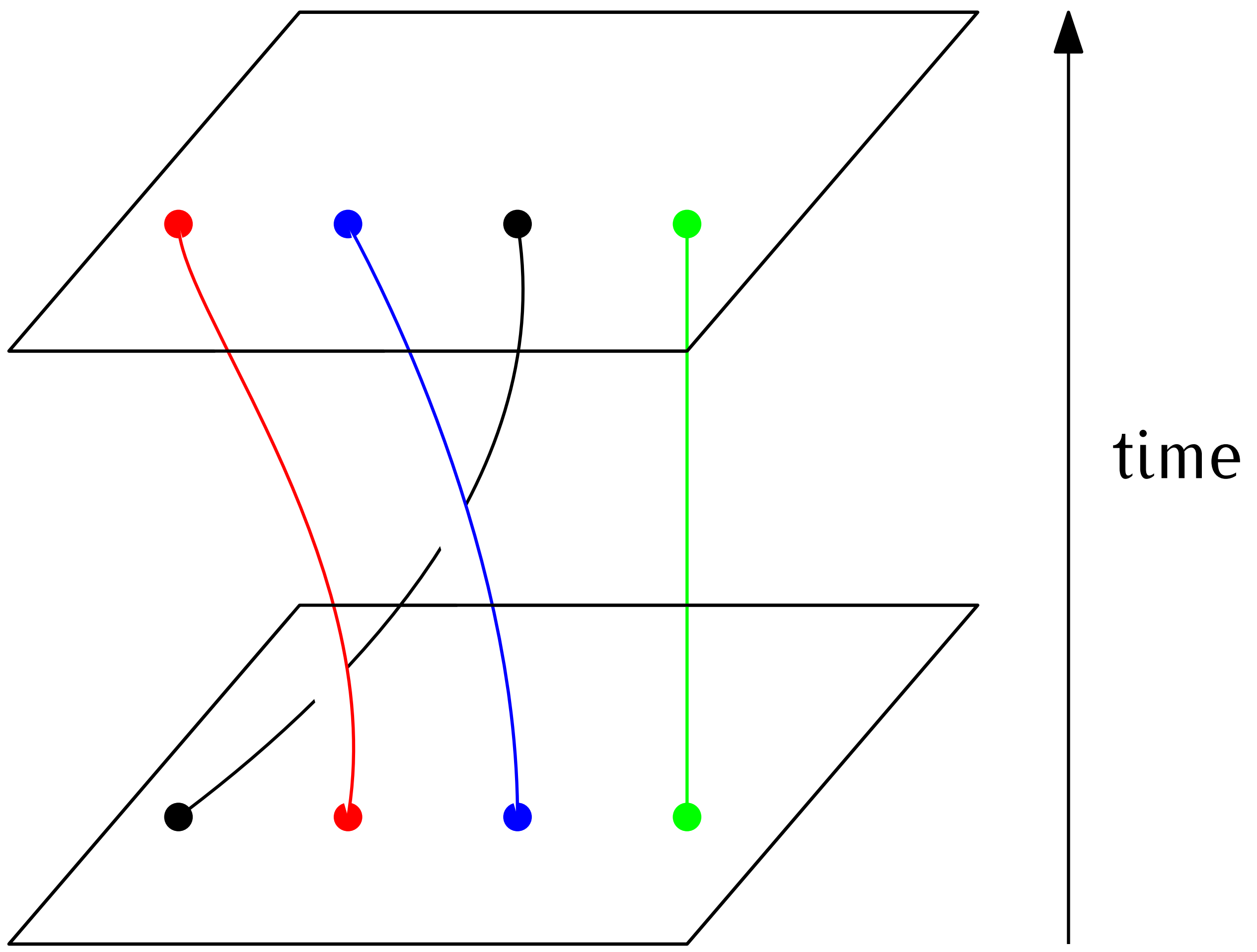

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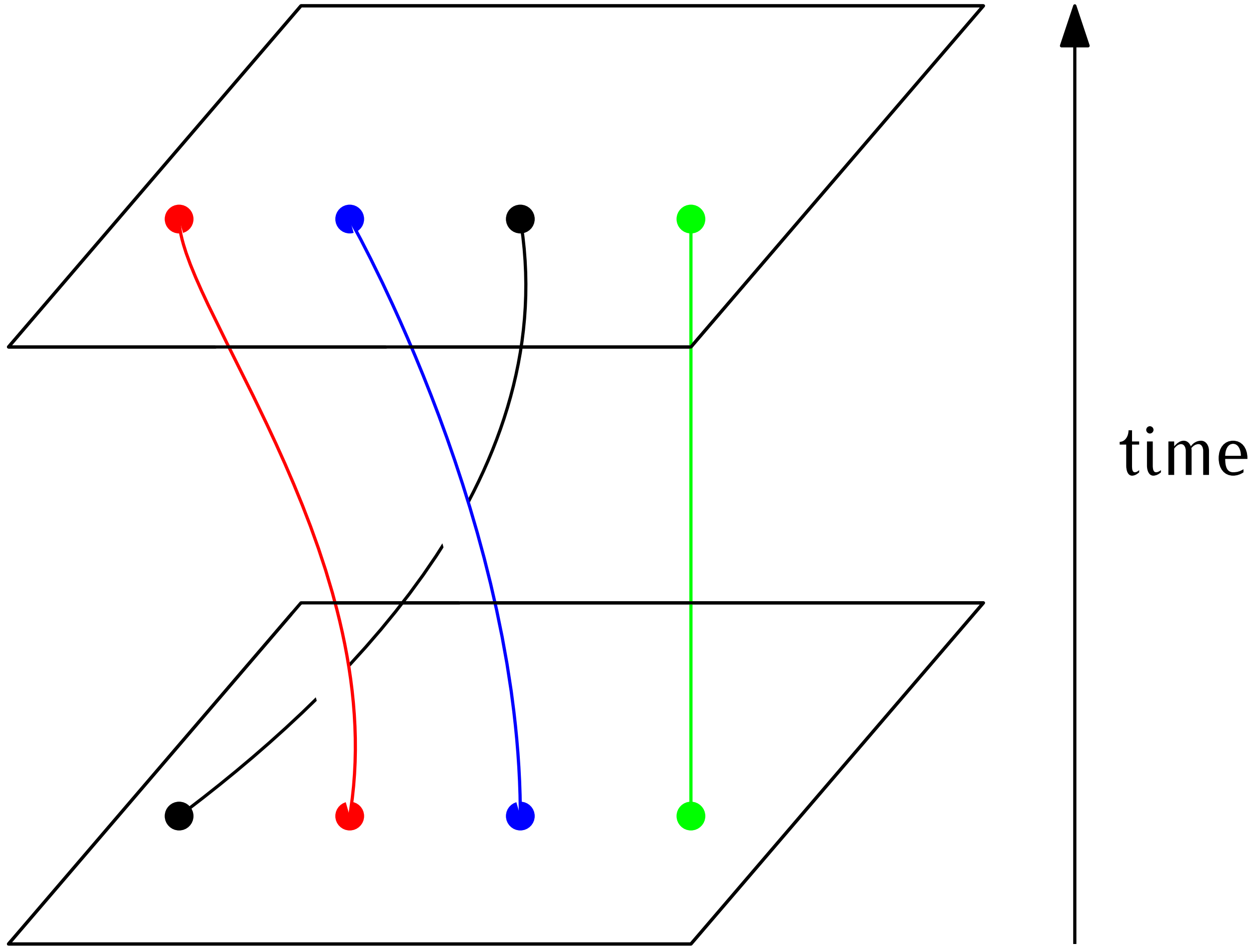

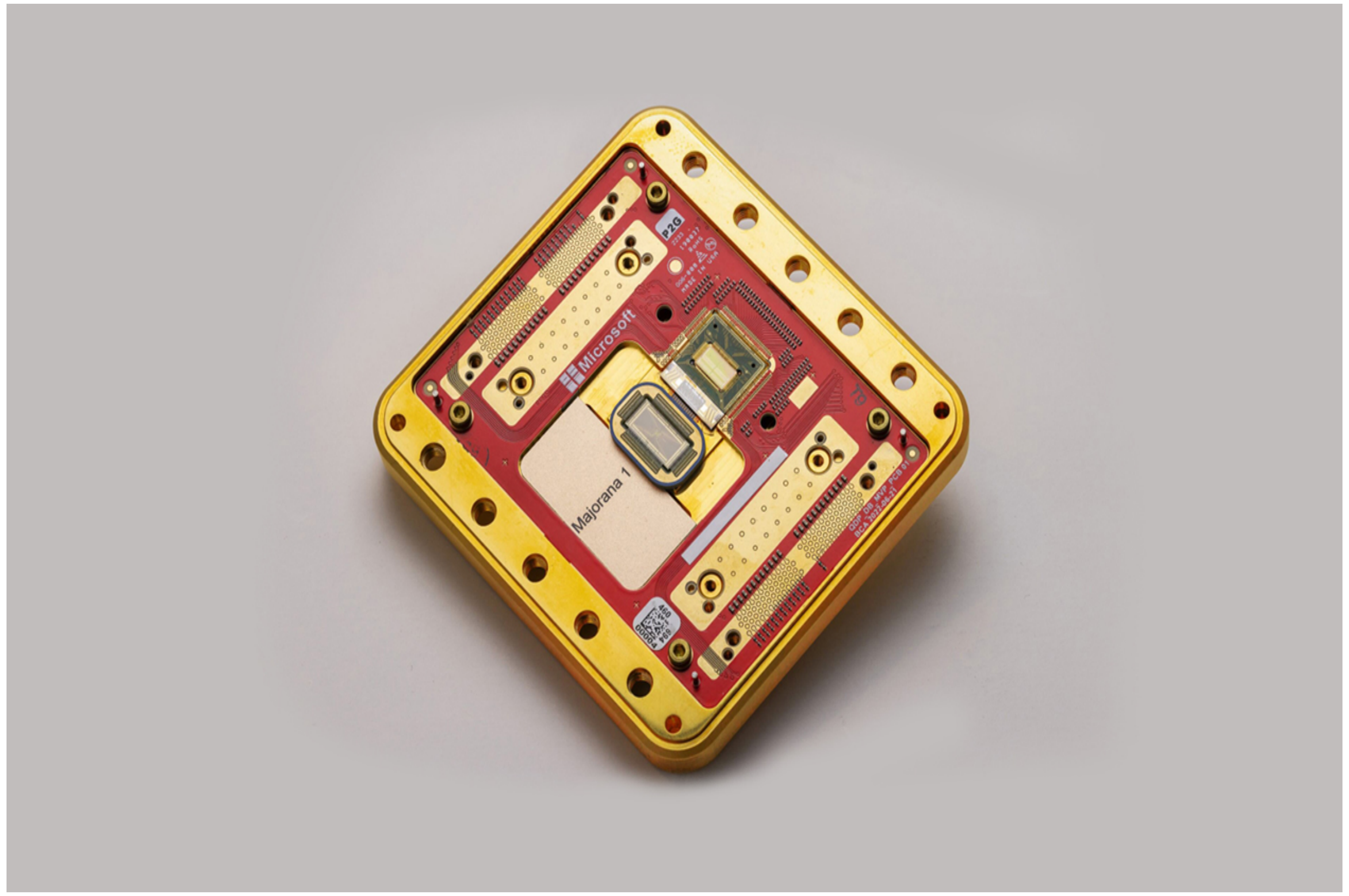

The bad news

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- #P-hard: for a given term, **counting** how many assignments make the term TRUE.

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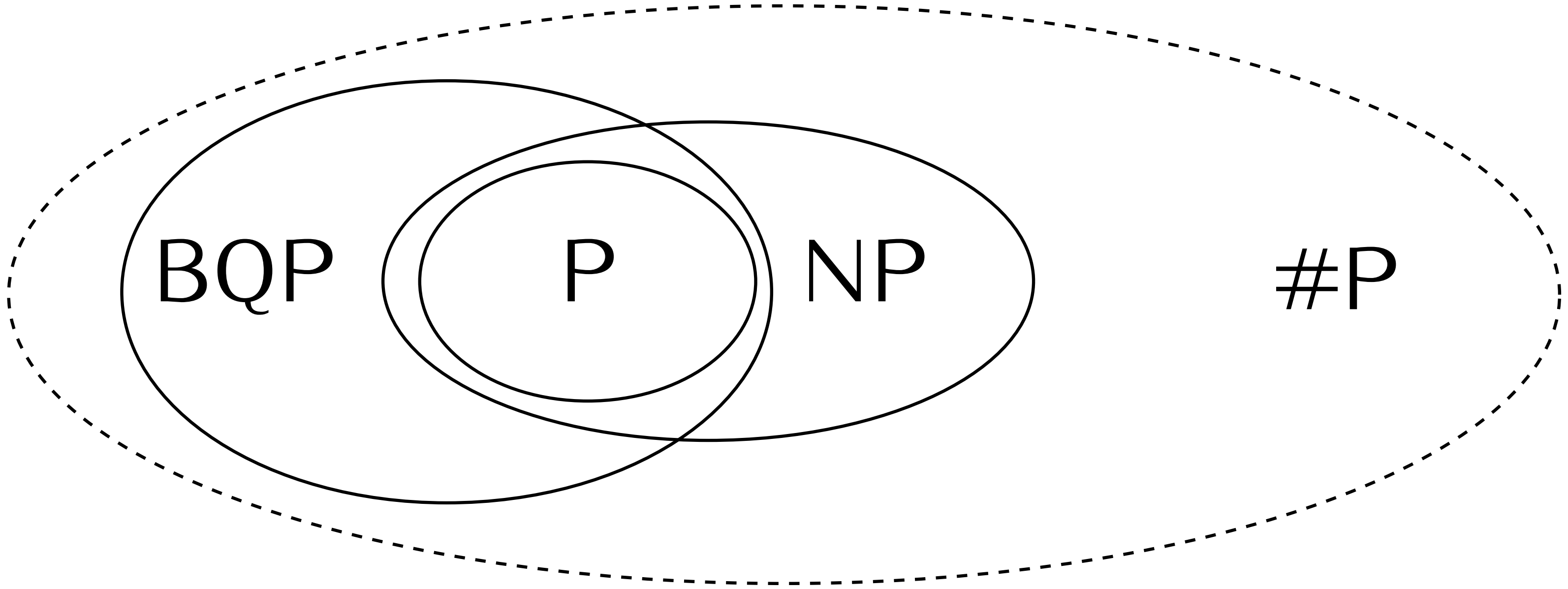
$$T : (x \land \neg y) \lor z$$

- P: given an assignment, telling whether the assignment gives a TRUE statement.
- NP-hard: for a given term, telling whether there exists **one** assignment that makes the sentence TRUE.
- #P-hard: for a given term, **counting** how many assignments make the term TRUE.



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Merci!